AVDTC of Generalized 3-Halin Graphs

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Abstract—In this paper, we prove that the adjacent vertex distinguishing total chromatic number of a generalized Halin graph with maximum degree 3 is 5. This result answers an open question of Chen and Zhang [11].

I. INTRODUCTION

Graph coloring is a fundamental topic in graph theory, which has a long history dating back to 1852 (see, e.g., [6] for a historical overview). The first combinatorial results deal almost exclusively with planar graphs in the form of coloring of maps. Since coloring a map is equivalent to coloring the vertices of its dual, vertex colorings have received most attention. The literature for graph colorings is vast, and the requirement of different restrictions lead to different types of colorings. The most important ones are outlined in the following.

Proper Vertex Coloring: Only the vertices of the graph are colored under the restriction that adjacent vertices have different colors.

Proper Edge Coloring: Only the edges of the graph are colored under the restriction that edges with common endpoints have different colors.

Proper Total Coloring: Both vertices and edges are colored so that adjacent or incident elements have different colors.

The minimum number of colors required in the above coloring types are called chromatic number, chromatic index and total chromatic number respectively.

The color set of a vertex is defined as the union of the colors of its incident edges and the color of itself (if any). The study of proper edge colorings that induce different color sets on vertices was introduced by Burris and Schelp [4] and Černý et al. [5], who examined the number of colors needed for proper edge colorings so that any two vertices have different color sets; such colorings are called vertex-distinguishing-proper-edge-colorings (or VD-edge-colorings, for short). Note that a VD-edge-coloring of a graph $G$ is also a proper edge coloring of $G$. By the well-known Vizing’s theorem [18], it follows that the chromatic index of a graph $G$ equals either $\Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ denotes the degree of $G$. However, for the VD chromatic index it is surprisingly difficult to formulate any such result, see, e.g., [1], [4].

Zhang et al. [24] slightly relaxed the restrictions implied by VD-edge-colorings and examined proper edge colorings that distinguish only pairs of adjacent vertices, thus introducing adjacent-strong-edge-colorings or adjacent-vertex-distinguishing-proper-edge-colorings (AVD-edge-coloring, for short). In their paper, Zhang et al. [24] determined the AVD chromatic index of several classes of graphs and they conjectured that the AVD chromatic index of a graph $G$ is at most $\Delta(G) + 2$. Balister et al. [2] positively answered the conjecture for graphs with maximum degree 2 and bipartite graphs.

The extension of AVD-edge-colorings to total colorings was made independently by Chen [8] and Zhang et al. [23], who introduced adjacent-vertex-distinguishing-total-colorings (AVDTC, for short), see Figure 1. In their work, Zhang et al. determined the AVDTC chromatic number of several classes of graphs, such as cycles, complete graphs, complete bipartite graphs, fans, wheels and trees. They also conjectured that the AVDTC chromatic number of a graph $G$ is upper bounded by $\Delta(G) + 3$. For the case where $\Delta(G) = 3$ the AVDTC conjecture has been positively answered independently by Wang [19] and Chen [9]. Hulgan [16] presented a more concise proof of this result. In fact, Hulgan showed that for a graph $G$ with $\Delta(G) = 3$, there exists an adjacent-vertex-distinguishing-total-coloring with 6 colors with the additional property that at most one color appears on both edges and vertices (refer to Figure 1a). In 2012, Coker and Johannson [14] used a probabilistic analysis to derive an upper bound of $\Delta(G) + O(1)$, while Huang et al. [15] proved that the AVDTC chromatic number is at most $2\Delta(G)$.

Exact values on the AVDTC chromatic number have been given for particular classes of graphs: the graph obtained by the $K_{2n+1}$ after the deletion of two adjacent edges [8], connected graphs with only one cycle and the square of cycles [12], the Mycielski graphs of paths, cycles, complete graphs, complete bipartite graphs, stars, fans and wheels [13]. Related results are also given in [3], [7], [11], [17], [22].

For outerplanar graphs, Chen and Zhang [10] examined the AVDTC chromatic number of 2-connected outerplanar graphs with $\Delta(G) \leq 4$. However, the AVDTC chromatic number of outerplanar graphs was fully determined by Wang and Wang [22]: the AVDTC chromatic number of outerplanar graphs is at most $\Delta(G) + 2$ in general, while equality holds only if there are adjacent vertices of maximum degree. The same bounds hold for planar graphs with large maximum degree [20].

Although for complete graphs of odd order the AVDTC conjecture’s upper bound is tight [23], it can be reduced by 1 for several classes of graphs: for outerplanar graphs [22] and planar graphs of large maximum degree [20] (as already mentioned), for cycles, paths, trees, fans, wheels, complete graphs of even order and complete bipartite graphs [23], for connected graphs with one cycle and $C_n^2$ for $n \geq 6$ [12], for the graph $P_n^k$ where $k = 2, 3, 4$ and $n \geq 6$ [17], for generalized Halin graphs with $\Delta(G) \geq 5$ [11], for graphs with small maximum average degree [21], for the hypercube $Q_n$ [7] and the generalized hypercube $K_n^d$ [3]. In this regard, Hulgan [16] asks whether the bound of 6 for graphs of degree 3 is in general tight.
II. AVDT-colorings of Generalized Halin Graphs with $\Delta = 3$

A Halin graph $G$ is a plane graph such that: (i) the vertices of the outerface $f_0$ have degree 3, (ii) after removing all edges of $f_0$ the remaining subgraph of $G$ is a tree $T$, called the backbone of $G$, whose interior vertices have $\delta \geq 3$. If we allow $T$ to contain vertices of degree two, $G$ is a generalized Halin graph; see Figure 2a for an example.

Chen and Zhang [11] showed that if $G$ is a generalized Halin graph with $\Delta(G) \geq 6$ and no two adjacent vertices are of maximum degree, then its AVDT chromatic number equals $\Delta(G) + 1$; if adjacent vertices of maximum degree are allowed and $\Delta(G) \geq 5$, then the AVDT chromatic number of $G$ is $\Delta(G) + 2$. The same bounds hold for planar graphs with $\Delta \geq 14$ [20]. Hence, the cases of graphs with small maximum degree are open.

In this section, we prove that the AVDT chromatic number of a generalized Halin graph $G$ with $\Delta(G) = 3$ is $\Delta(G) + 2 = 5$. This result strengthens both previously mentioned results and also answers an open question of Chen and Zhang [11]. Also, it is a step towards a positive answer to the question posed by Hulgan [16], whether 5 colors suffice for the AVDT-coloring of graphs with $\Delta(G) = 3$. Since $\Delta(G) = 3$, the backbone $T$ of $G$ has inner vertices (i.e., vertices that are not leaves in $T$) of degree 2 or 3. Note that since all vertices of $f_0$ have degree exactly 3, it follows that there exist at least two adjacent vertices of $G$ that are of maximum degree. This in conjunction with the result of Zhang et al. [23], who proved if $G$ has two vertices of maximum degree which are adjacent, then its AVDT chromatic number is at least $\Delta(G) + 2$, implies that our bound of 5 is optimal.

Consider now a tree $T$ of maximum degree 3. Based on $T$, we construct another tree $T_C$ as follows. Let $P = u_0, \ldots, u_s, u_{s+1}$ be a path in $T$ of length $s + 2$, where $s \geq 2$, such that all vertices of $P$ have degree 2 except for its endpoints $u_0$ and $u_{s+1}$ that have degree 1 or 3. We contract $P$ to a shorter path of length 2 or 3 depending on the parity of $s$. In particular, if $s$ is even, then we replace $P$ by the edge $(u_0, u_{s+1})$, and if $s$ is odd we replace $P$ by the path $P_C = u_0, u, u_{s+1}$. We apply the above contraction operation for every such path $P$ of $T$. We say that the produced tree $T_C$ is the contracted tree of $T$ (see Figure 2b). It is not hard to see that $T_C$ is uniquely derived from $T$, $T$ is a subdivision of $T_C$ and that $T_C$ has the same number of leaves as $T$ and the same number of vertices of degree 3. Also, if there is a vertex $v$ with $d_{T_C}(v) = 2$, then both neighbors of $v$ in $T_C$ have degree 1 or 3. If $T_C$ is the same tree as $T$, we say that $T$ satisfies the contraction property.

Generalized Halin graphs satisfying the contraction property. We will first prove our bound for generalized Halin graphs whose backbone trees satisfy the contraction property and then we will generalize our approach to all generalized Halin graphs with maximum degree 3. So, in the following let $T$ be a tree that satisfies the contraction property and let $G$ be the generalized Halin graph with $T$ as its backbone. Our proof is by induction on the number of vertices of $T$. The base of the induction will correspond to one of the following cases:

- If $T$ has three vertices, then $T$ is isomorphic to path $P_3$ (see Figure 3a).
- If $T$ has four vertices, then $T$ is isomorphic to star $K_{1,3}$ (see Figure 3b).
- If $T$ has five vertices, then $T$ is isomorphic to the tree of Figure 3c.
- If $T$ has six vertices, then $T$ is isomorphic either to the tree of Figure 3d or to the tree of Figure 3e.

Note that the trees of Figures 3a-3b (3c-3d) are the only trees with $\Delta = 3$ that satisfy the contraction property and have diameter 2 (3, respectively). For a tree $T$ that satisfies the contraction property and has more than 6 vertices, let $u$ and $v$ be two peripheral vertices of $T$, i.e., $u$ and $v$ are leaves of $T$ so that $d(u, v) = diam(T)$. Let also $P_{uv}$ be the path in $T$ connecting $u$ and $v$. If $T$ has more than 6 vertices, then let $w$ be the vertex of $P_{uv}$ with $d(u, w) = 2$ and $y$ the vertex of $P_{uw}$ with $d(u, y) = 3$. Note that in this case $diam(T) \geq 4$. Hence, vertices $u$, $w$, and $y$ are distinct. We distinguish two cases depending on whether $w$ has degree 2 or 3 in $T$.

- If $d(w) = 2$, the contraction property gives that the degree of the neighbors of $w$ is either 1 or 3. In this case it should be always 3 since $w$ has distance 2 from $u$, which is a leaf, and since $T$ has diameter at least 4. Let $x$ be the neighbor of $w$ with $d(u, x) = 1$. Since $u$ is a peripheral vertex of $T$ the third neighbor of $x$ (that is not on $P_{uw}$) is a leaf of $T$ (see Figure 4a).
- If $d(w) = 3$, then we have a total of 6 different cases that are shown in Figures 4b-4g. These cases correspond to all rooted trees where the root $w$ has two children, the height of the tree is 2 and the maximum degree is 3.
The base of the induction. The base of the induction is formed by all generalized Halin graphs, whose backbone trees have at most 6 vertices. In Figures 3f-3j, we illustrate valid AVDT-colorings for these graphs. Note that all graphs are simple, except for the one of Figure 3f, which contains a double edge (required for the induction step, as we will shortly see).

The inductive step. Assume that all generalized Halin graphs that (i) are of maximum degree 3, (ii) have \( n_0 > 6 \) vertices and (iii) whose backbone trees satisfy the contraction property, have AVDT chromatic number 5. We will prove that if the backbone \( T \) of a generalized Halin graph \( G \) has \( n > n_0 \) vertices, \( \Delta(G) = 3 \) (and satisfies the contraction property), then its AVDT chromatic number is also equal to 5. From our analysis, it follows that \( T \) has two peripheral vertices \( u \) and \( v \) with \( d(u, v) \geq 4 \) and \( T \) is one of the trees shown in Figure 4.

The main idea is that for each one of these cases, we will prune \( T \) to a smaller tree \( T' \) that satisfies the induction hypothesis. In particular, \( T' \) is obtained from \( T \) by removing from it the subtree, which is rooted at \( w \) and contains \( u \) (without, however, removing vertex \( w \); refer to the parts of \( T \) in the dashed drawn regions of Figure 4). In this way, \( T' \) contains vertex \( v \) (but not \( u \)) and has \( w \) as a leaf (recall that \( w \) is at distance 2 from \( u \) in \( T \)). As a subtree of \( T \), \( T' \) satisfies the contraction property and has \( n' \geq n - 6 \) vertices. Also note that \( n' \geq 3 \) since \( \text{diam}(T') \geq \text{diam}(T) - 2 \geq 2 \). So, by the induction hypothesis, the generalized Halin graph \( G' \) with backbone \( T' \) has an AVDT-coloring with 5 colors.

Without loss of generality, assume that the produced coloring of \( T' \) is the one shown in Figure 5, where \( w_1 \) and \( w_2 \) are the two neighbors of \( w \) along the outerface \( f_0 \) of \( G' \). Note that \( w_1 \rightarrow w_2 \) may occur, but only in the case where \( G' \) is the graph of Figure 3f. Edges \((w, w_1)\), \((w, w_2)\) and \((w, y)\) have colors \(1\), \(2\) and \(3\) respectively, while vertex \( w \) has color \(4\). So, the only color that does not belong to the color set of \( w \) is color \(5\). For simplicity, if a vertex of degree \(3\) has color \(p\) and \(c\) is the only color not in its color set, we say that the color of the vertex is \(p(c)\). Similarly, if a vertex of degree \(2\) has color \(p\) and \(c_1, c_2\) are the two colors that do not belong to its color set, we say that the color of the vertex is \(p(c_1, c_2)\). Following this notation, \(G\) and \(G'\) have \(c_1\) and \(c_2\) as the colors for \(p\) and \(c\), respectively.

- Let \( z \) be a vertex of degree \(2\) with incident edges \((z, z_1)\) and \((z, z_2)\), where \(z_1\) and \(z_2\) have degree \(3\) in \(G\) by the contraction property. Suppose that \((z, z_1)\) has color \(p_1(c)\) and \((z, z_2)\) has color \(p_2(c)\) (where \(p_i \neq c_i, i = 1, 2\)). There are three available colors for \(z\) and at least one of them is different from \(p_1\) and \(p_2\). Also, the color set of \(z\) consists of three colors and is different from the color sets of its two neighbors \(z_1\) and \(z_2\), since they contain \(4\) colors. This implies that in order to produce a valid AVDT-coloring for \(G\), it suffices to color all uncolored edges and vertices of degree \(3\).
- Let \(z_1\) and \(z_2\) be two adjacent vertices of degree \(3\), whose incident edges are colored. There are two colors available for \(z_1\), say \(\{p_1, p'_1\}\) and two colors for \(z_2\), say \(\{p_2, p'_2\}\). Then, \(z_1\) can take color \(p_1(p'_2)\) or \(p'_1(p_2)\); \(i = 1, 2\). If the two sets are disjoint, i.e., \(\{p_1, p'_1\} \cap \{p_2, p'_2\} = \emptyset\), then the coloring of \(z_1\) is independent from the coloring of \(z_2\) in the sense that any coloring distinguishes them.
Assume that $T$ is the tree shown in Figure 4a. Since $w$ has degree 2 in $T$, $y$ has degree 3 by the contraction property and color $p_3(c_3)$. By the coloring of $G'$ we have a partial coloring of $G$ (refer to Figure 6a): it remains to extend it to an AVDT-coloring of $G$. We color edges $(u, u')$, $(u, x)$, $(u', x)$ and $(x, w)$ with colors $3$, $4$, $5$ and $1$ respectively, and vertex $w$ with color $4(2,5)$. For vertex $u$ the only available colors are $\{2,5\}$, for vertex $x$ colors $\{2,3\}$ and for vertex $u'$ colors $\{1,4\}$ (see Figure 6b). Note that $\{1,4\} \cap \{2,5\} = \emptyset$ and $\{1,4\} \cap \{2,3\} = \emptyset$, which implies that the coloring of $u'$ is independent from the coloring of its neighbors $u$ and $x$. So, starting from vertex $u'$, there are two possible colorings $4(1)$ and $1(4)$. Since $p_2 \neq 4$, vertex $u'$ cannot be colored as $4(1)$ if and only if $c_2 = 1$. But in this case, we can use the second coloring $1(4)$, since $p_2 \neq c_2 = 1$ and $c_2 \neq 4$. Hence, vertex $u'$ can always be colored. Now for the other two vertices, there are two possible colorings:

$$
\begin{align*}
&\{ u : 2(5) \text{ and } x : 3(2) \} \\
&\{ u : 5(2) \text{ and } x : 2(5) \}
\end{align*}
$$

With respect to vertex $u$, the coloring $2(5)$ is not applicable if and only if $p_1 = 2$ (since $c_1 \neq 5$). But in this case, the second coloring $5(2)$ is possible. Hence, in any case the coloring of $G'$ can be extended to a proper AVDT-coloring of $G$.

Let now $T$ be the tree of Figure 4b. In Figure 6c we can see the partial coloring of $G$ produced by the coloring of $G'$. We color edges $(u, u')$, $(u, x)$, $(u', x)$ and $(x, w)$ with colors $4$, $3$, $5$ and $1$ respectively. For vertex $u$ the only available colors are $\{2,5\}$, for vertex $u'$ colors $\{1,3\}$, for vertex $w$ colors $\{2,4\}$ and for vertex $x$ colors $\{1,4,5\}$ (see Figure 6d). If one considers only vertices of degree 3, the coloring of $u$ depends only on the coloring of $w_1$, the coloring of $u'$ on the one of $w_2$ and the coloring of $w$ on the coloring of $y$ (this is because the sets of available colors of uncolored adjacent vertices are disjoint). So, for vertex $u$ there are two possible colorings $2(5)$ and $5(2)$. Since $c_1 \neq 5$, $u$ cannot be colored as $2(5)$ if and only if $p_1 = 2$. But in this case we can use the second coloring $5(2)$. Hence, $u$ can always be colored. The same applies for vertex $u'$ and $w$, if $d(y) = 3$. However, if $y$ is of degree 2 with color $p_3(c_3', c_3')$, then $w$ can always take color $4(2)$, since $p_3 \neq 4$. In this scenario, the two vertices have different degree and therefore different color sets. The only vertex that remains uncolored is $x$. Since $x$ has degree 2, it can always be properly colored. So, in any case the coloring of $G'$ can be extended to a proper AVDT-coloring of $G$.

If $T$ is the tree of Figure 4c, then the partial coloring of $G$ produced by the coloring of $G'$ is the one shown in Figure 6e. We color all uncolored edges and vertex $w$ as in Figure 6f. Considering only vertices of degree 3, the coloring of $u$ depends only on the coloring of $w_1$ and the coloring of $u'$ on the one of $w_2$. So, for vertex $u$ there are two possible colorings $4(2)$ and $2(4)$. Since $p_1 \neq 4$, $u$ cannot be colored with $4(2)$ if and only if $c_1 = 2$. But in this case we can use the second coloring $2(4)$. Hence, $u$ can always be colored. The same applies for vertex $u'$. The only vertices that remain uncolored are $x$ and $x'$, which however are of degree 2. Hence, they can easily be colored, which implies that the coloring of $G'$ can be extended to a proper AVDT-coloring of $G$.

Next, assume that $T$ is the tree of Figure 4d. The partial coloring of $G$ implied by the coloring of $G'$ is shown in Figure 6g. We color all uncolored edges and $u'$ as in Figure 6h. We first examine the vertices that are of degree 3, i.e., $u$, $w$, $x'$ and $u''$. The coloring of $u$ depends only on the coloring of $w_1$ and so we can always find a proper coloring for $u$. For the remaining vertices of degree 3 we have two possible colorings:

$$
\begin{align*}
&\{ w : 1(2), \ x' : 2(3) \text{ and } u'' : 1(2) \} \\
&\{ w : 2(1), \ x' : 3(2) \text{ and } u'' : 2(1) \}
\end{align*}
$$

Since the coloring of $w$ may be determined by the coloring of $y$, the colorings of $x'$ and $u''$ are also determined. However, the coloring of $x'$ is independent from the coloring of its third neighbor $w_2$ and the coloring of $u''$ is also independent from the coloring of $w_2$ and $u$. Finally, vertex $x$ has degree 2 and hence it can be always properly colored giving rise to a proper AVDT-coloring of $G$.

We continue with the case where $T$ is the tree of Figure 4e. Figure 6i shows the corresponding partial coloring of $G$. In Figures 6j and 6k we can see two colorings of $G$, where vertex $u$ has color $4(5)$ (recall that $p_1 \neq 4$ and $c_1 \neq 5$) and:

$$
\begin{align*}
&\{ u' : 4(3) \text{ and } w : 2(5) \} \\
&\{ u' : 3(5) \text{ and } w : 4(2) \}
\end{align*}
$$

Suppose that none of the above colorings is valid, i.e., $u'$ is not distinguished from $w_2$ and/or $w$ is not distinguished from $y$. We consider two cases depending on color $c_2$.

1. $c_2 = 3$: In this case, the first coloring is not applicable as $u'$ would have the same color set as $w_2$. If, on the other hand, we use color $3(5)$ for $u'$, it is distinguished from $w_2$, since $p_2 \neq 3 = c_2$ and $c_2 \neq 5$. Under the assumption that the second coloring is not valid either, it follows that $w$ is not distinguished from $y$. For vertex $y$, we have that $p_3 \neq 4$. So, $y$ is a vertex of degree three with $c_3 = 2$.

2. $c_2 = 3$: In this case, the first coloring distinguishes $u'$ from $w_2$ (since $p_2 \neq 4$). Hence, $w$ is not distinguished from $y$. This can only occur, if $p_3 = 2$. But then, the second coloring distinguishes $w$ from $y$. Hence, $u'$ cannot be colored as $3(5)$ and therefore $p_2 = 3$.

So, the two colorings are not valid, if (i) $c_2 = 3$ and $y$ is a degree-3 vertex with $c_3 = 2$, or (ii) $p_2 = 3$ and $p_3 = 2$. For the first case, we use the coloring of Figure 6l and for the second case the one of Figure 6m. It follows that in both cases, it is possible to produce a proper AVDT-coloring of $G$, completing the case where $T$ corresponds to the tree of Figure 4e.

The case where $T$ is the tree of Figure 4f is symmetric to the case of Figure 4d. So, the last case to consider is the one
where $T$ is the tree of Figure 4g. The corresponding partial coloring of $G$ is shown in Figure 6a. In Figure 6a, we can see a coloring of $G$, where vertices $u$, $u'$ and $w$ are color with 4(5), which is clearly a proper AVDT-coloring for $G$. Summarizing all the above, we have proved the following lemma:

**Lemma 1:** Let $G$ be a generalized Halin graph with maximum degree 3, whose backbone $T$ satisfies the contraction property. Then, the AVDT chromatic number of $G$ is 5.

It remains to extend Lemma 1 for generalized Halin graphs $G$ with backbone $T$ that does not necessarily satisfy the contraction property. Let $T_C$ be the contracted tree of $T$ and $G_{C}$ the generalized Halin graph with backbone $T_C$. Let $P = u_0, u_1, \ldots, u_s, u_{s+1}$ be a path in $T_C$ of length $s+2$, where $s \geq 2$, such that all vertices of $T$ have degree 2 except for its endpoints $u_0$ and $u_{s+1}$ that have degree 1 or 3. We distinguish two cases based on the parity of $s$.

If $s$ is even, then in $T_C$, path $P$ is contracted to a single edge $(u_0, u_{s+1})$, where vertices $u_0$ and $u_{s+1}$ have degree 3 in $G_{C}$. Suppose that there is an AVDT-coloring of $G_{C}$ with 5 colors. Let $p$ be the color of edge $(u_0, u_{s+1})$, $p_1(c_1)$ the color of $u_0$ and $p_2(c_2)$ the color of $u_{s+1}$, with $p \neq p_1, p_2, c_1, c_2$ (refer to Figure 7a). Let $p'$ be a color that is different from $p$, $p_1$ and $p_2$ ($p'$ always exists). Then, we replace edge $(u_0, u_{s+1})$ with $P$ without changing the colors of its endpoints: edges $(u_0, u_1)$ and $(u_s, u_{s+1})$ have color $p$. We color $u_i$ with color $p_1$ if $i$ is even and with color $p_2$ otherwise, $1 \leq i \leq s$. For the edges, we color $(u_1, u_{s+1})$ with color $p$, if $i$ is odd and with color $p'$ otherwise, $1 \leq i \leq s - 1$ (note that $(u_{s-1}, u_s)$ takes color $p'$, different from the one of $(u_s, u_{s+1})$). Consider now two adjacent vertices of degree 2 in $P$. Their incident edges have colors $p$ and $p'$. One of the two vertices is colored with $p$ and has color $p_2$ in its color set, while the other vertex is colored $p_2$ and has color $p_1$ in its color set. Hence, they are distinguished. Vertices $u_0$ and $u_{s+1}$ are also distinguished from $u_1$ and $u_s$, respectively: $u_1$ has vertex-color $p_2$ and $u_s$ has color $p_1$, since $s$ is even. Also they have different degrees and therefore different color sets.

If $s$ is odd, then in $T_C$, path $P$ is contracted to the path $P_C = u_0, u, u_{s+1}$, where vertex $u$ has degree 2 in $G_{C}$ and vertices $u_0$ and $u_{s+1}$ have degree 3. Suppose that there is an AVDT-coloring of $G_{C}$ with 5 colors. Let $p$ be the color of
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Fig. 7: $P$ has $s + 2$ vertices in $T$, when $s$ is even; (a) coloring of $(u_0, u_{s+1})$ in $G_C$, (b) coloring of $P$ in $G$.

Fig. 8: $P$ has $s + 2$ vertices in $T$, when $s$ is odd; (a) coloring of $u_0, u, u_{s+1}$ in $G_C$, (b) coloring of $P$ in $G$.

After expanding all contracted paths one by one, the produced graph is $\tilde{G}$ and therefore we can state our main result.

**Theorem 1:** The AVDT chromatic number of a generalized Halin graph with maximum degree 3 is 5.

**III. Conclusions**

In this paper, we proved that the AVDT chromatic number of a generalized Halin graph with maximum degree 3 is 5, which answers an open question of [11]. Our results mainly leave the following two open questions: (i) is the AVDT chromatic number of a generalized Halin graph with maximum degree four equal to 6? (ii) is the bound of 6 tight for graphs of maximum degree 3 or are 5 colors always sufficient?

**References**


